# No-arbitrage implies rough volatility

Mathieu Rosenbaum

École Polytechnique

Cours Bachelier 2019

# Table of contents

# Properties of the rough volatility models

## Statistical analysis of rough volatility models

- The log-volatility behaves essentially as a fractional Brownian motion with Hurst parameter of order 0.1.
- More precisely, basically all the statistical stylized facts of volatility are retrieved when modeling it by a rough fractional Brownian motion.
- Such model also enables us to reproduce very well the behavior of the implied volatility surface, in particular the at-the-money skew (without jumps).
- Also very relevant for risk management of derivatives (closed form formulas, see for example the rough Heston model).
- The phenomenon is universal.

# In this presentation

**What we want to understand :**

- Why is volatility rough ?
- Something universal in finance $\rightarrow$ should be related to some no arbitrage concept.
- Can we make this link ?

# Table of contents

# Market impact

## Some definitions

- Market impact is the link between the volume of an order (either market order or metaorder) and the price moves during and after the execution of this order.
- We focus here on the impact function of metaorders, which is the expectation of the price move with respect to time during and after the execution of the metaorder.
- We call permanent market impact of a metaorder the limit in time of the impact function (that is the average price move between the start of the metaorder and a long time after its execution).

# Market impact in practice, from Lillo et al.

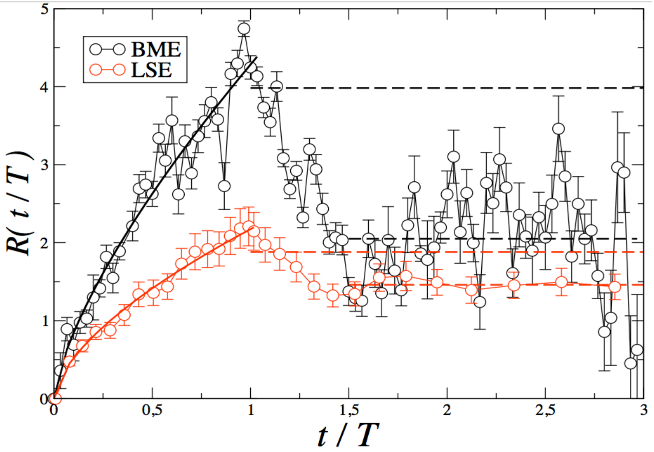


FIGURE : Market impact curves.

# Market impact

## Linear permanent impact

- Let $P_t$ be the asset price at time $t$. Consider a metaorder with total volume $V$.
- $$PMI(V) = \lim_{s \to +\infty} \mathbb{E}[P_s - P_0 | V].$$
- Price manipulation is a roundtrip with negative average cost.
- From Huberman and Stanzl and Gatheral : Only linear permanent market impact can prevent price manipulation : $PMI(V) = kV$.

# Market impact

## Linear permanent impact

- Let $P_t$ be the asset price at time $t$. Consider a metaorder with total volume $V$.
- $$PMI(V) = \lim_{s \to +\infty} \mathbb{E}[P_s - P_0 | V].$$
- Price manipulation is a roundtrip with negative average cost.
- From Huberman and Stanzl and Gatheral : Only linear permanent market impact can prevent price manipulation : $PMI(V) = kV$.

# Market impact

## CAPM like argument for linear permanent impact

- $n$ investors in the market. Two dates : $t = 0$ and $t = 1$.
- $N$ shares spread between the agents, price $P$ for the asset.
- Every investor $i$ estimates that the law of the price at time 1 has expectation $E_i$ and variance $\Sigma_i$. He chooses his number of asset $N_i$ such that

$$N_i = \text{argmax}_x [x(E_i - P) - \lambda_i x^2 \Sigma_i].$$

- We get

$$N_i = \frac{E_i - P}{2\lambda_i \Sigma_i}.$$

# Market impact

## CAPM like argument for linear permanent impact

- Since $\sum_{i=1}^n N_i = N$, we deduce

$$P = \frac{\sum_{i=1}^n \frac{E_i}{2\lambda_i \Sigma_i} - N}{\sum_{i=1}^n \frac{1}{2\lambda_i \Sigma_i}}.$$

- Let us now assume that the total number of shares becomes $N - N_0$ due to the action of some non-optimizing agent needing to buy some shares (for cash flow reasons for example). The new indifference price is

$$P^+ = P + \frac{N_0}{\sum_{i=1}^n \frac{1}{2\lambda_i \Sigma_i}} = P + kN_0.$$

# Dynamics

## Assumptions

- All market orders are part of metaorders.
- Let $[0, S]$ be the time during which metaorders are being executed (which can be thought of as the trading day). Let $v_i^a$ (resp. $v_i^b$) be the volume of the $i$-th buy (resp. sell) metaorder and $N_S^a$ (resp. $N_S^b$) be the number of buy (resp. sell) metaorders up to time $S$. Finally, write $V_S^a$ and $V_S^b$ for cumulated buy and sell order flows up to time $S$.
- We assume

$$P_S = P_0 + k\Big(\sum_{i=1}^{N_S^a} v_i^a - \sum_{i=1}^{N_S^b} v_i^b\Big) + Z_S = P_0 + k(V_S^a - V_S^b) + Z_S,$$

with $Z$ a martingale term that we neglect.

# Dynamics

## Martingale assumption

- We furthermore assume that the price $P_t$ is a martingale. We obtain

$$P_t = P_0 + \mathbb{E}\big[k(V_S^a - V_S^b)|F_t\big].$$

- We suppose that $\lim_{S \to +\infty} \mathbb{E}\big[k(V_S^a - V_S^b)|F_t\big]$ is well defined. This means

$$\mathbb{E}\big[(V_{S+h}^a - V_{S+h}^b) - (V_S^a - V_S^b)|F_t\big] \to 0,$$

  that is the order flow imbalance between $S$ and $S+h$ is asymptotically (in $S$) not predictable at time $t$.

# Dynamics

## Price dynamics

- Under the preceding assumptions, we finally get

$$P_t = P_0 + k \lim_{S \to +\infty} \mathbb{E}\big[(V_S^a - V_S^b)|F_t\big].$$

- Martingale price.
- Linear permanent impact, independent of execution mode.
- The price process only depends on the global market order flow and not on the individual executions of metaorders. We thus do not need to assume that the market sees the execution of metaorders as it is usually done.
- Market orders move the price because they change the anticipation that market makers have about the future of the order flow.

# Table of contents

# Preliminary : Hawkes processes

## Hawkes process

- A Hawkes process $(N_t)_{t\geq 0}$ is a self exciting point process, whose intensity at time $t$, denoted by $\lambda_t$, is of the form

$$\lambda_t = \mu + \sum_{0<J_i<t} \phi(t - J_i) = \mu + \int_{(0,t)} \phi(t-s)dN_s,$$

  where $\mu$ is a positive real number, $\phi$ a regression kernel and the $J_i$ are the points of the process before time $t$.
- These processes have been introduced in 1971 by Hawkes in the purpose of modeling earthquakes and their aftershocks. First introduction in finance : Chavez-Demoulin *et al.* (2005), Bowsher (2007).

# Hawkes specification

## Hawkes propagator

- We now assume that buy and sell order flows are modeled by independent Hawkes processes $N^a$ and $N^b$ with same parameters $\mu$ and $\phi$. All orders have same unit volume.
- Later on we will consider an asymptotic setting so that the flows are defined on $[0, T]$ with $T \to +\infty$.
- To be very general, we allow the parameters to depend on $T$ (but do not assume they do). So we write $N^{a,T}$, $N^{b,T}$, $\mu^T$, $\phi^T = a^T \phi$ with $a^T < 1$ and $\int \phi = 1$ (stability condition).
- Note that the average intensity of our processes is essentially $\beta^T = \mu^T (1 - a^T)^{-1}$ (stationary case).

# Price dynamic under Hawkes specification

## Price equation

- In this case, the general equation above rewrites as the following propagator dynamic

$$P_t = P_0 + \int_0^t \zeta^T(t-s)(dN_s^{a,T} - dN_s^{b,T}),$$

with $\zeta^T(t) = \big(1 + \int_t^{+\infty} \psi^T(u) - \int_0^t \psi^T(u-s)\phi^T(s)dsdu\big)$.

- The propagator kernel compensates the correlation of the order flow implied by the Hawkes dynamics to recover a martingale price. Note that the kernel does not tend to 0 since there is permanent impact.

# Adding our own transactions

## Labeled order

- In the above framework, $N^{a,T}$ and $N^{b,T}$ are the flows of anonymous market orders.
- Now assume we arrive on the market, executing our own (buy) metaorder. Our flow is a Poisson process $n$ on $[0, T]$ (can be generalized) with intensity $I^T = \gamma\beta^T$, $\gamma < 1$ (proportion $\gamma$ of the total flow).
- According to the propagator approach, we get

$$P_t = P_0 + \int_0^t \zeta^T(t-s)(dN_s^{a,T} - dN_s^{b,T}) + \int_0^t \zeta^T(t-s)dn_s.$$

# Impact function

## Explicit market impact

- We get that the impact function of a metaorder executed between 0 and $T$ is for $0 \leq t \leq T$

$$MI(t) := \mathbb{E}[P_t - P_0] = I^T \int_0^t \zeta^T(t-s)ds.$$

- We define

$$\overline{MI}^T(t) = \frac{MI_{tT}^T}{T\beta^T} = \int_0^t \chi^T(t-s)\mathrm{d}s,$$

with

$$\chi^T(s) = \gamma \frac{\zeta^T(Ts)}{1-a^T}.$$

# Decomposing the impact

## Transient and permanent market impact

- We have

$$\overline{MI}^T(t) = \int_0^t \chi^T(t-s)\mathrm{d}s,$$

$$\chi^T(s) = \gamma\big(1 + (1-a^T)^{-1}\int_{Ts}^{+\infty}\phi\big).$$

- The market impact kernel is the sum of a linear market impact representing the permanent component and of a transient term vanishing after the metaorder completion.
- Existence of transient part is equivalent (asymptotically) to the existence of a limit for $(1-a^T)^{-1}\int_{Ts}^{+\infty}\phi$.

# Shape of the market impact

## Power-law market impact

Assume the transient part of the market impact exists. Then for $t < 1$,

$$\lim_{T \to +\infty} \overline{MI}^T(t) - \gamma t = \gamma K t^{1-\alpha}$$

for some $K > 0$ and $\alpha \in (0,1)$. Furthermore, we necessarily have $a^T \to 1$ (highly endogenous market) and the tail of the Hawkes kernel is power-law of order $x^{-(1+\alpha)}$.

Note that the celebrated square-root law (Bouchaud et al., Farmer et al., Pohl et al.) corresponds to $\alpha = 1/2$.

# Limiting price process

## Emergence of (hyper-)rough processes

Let $\bar{P}^T_t = \frac{1}{T\beta^T} P^T_t$ and assume $\mu^T(1-a^T)T$ tends to $\delta$. As $T$ goes to infinity, the limit $P_t$ of $\bar{P}^T_t$ satisfies

$$P_t = B_{X_t},$$

$$X_t = \frac{2}{\delta}\int_0^t F^{\alpha,\lambda}(s)\mathrm{d}s + \frac{1}{\delta\sqrt{\lambda}}\int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}W_{X_s},$$

where $B$ and $W$ are Brownian motions, $\lambda = K\Gamma(1-\alpha)^{-1}$ and $F^{\alpha,\lambda}(t) = \int_0^t f^{\alpha,\lambda}(s)\mathrm{d}s$ with $f^{\alpha,\lambda}$ the density of the Mittag-Leffler distribution. Furthermore, $X$ has Hölder regularity $\min(2\alpha, 1) - \varepsilon$.

# Uniqueness in law of the limit

## Characterization of the limit

Let $X$ be the cumulated volatility process of the limiting price, $f \in C^0(\mathbb{R}^+, \mathbb{R}^-)$. The function $K(f,t) = \mathbb{E}[\exp\big(\int_0^t f(s)\mathrm{d}X_{t-s}\big)]$ satisfies

$$K(f,t) = \exp\Big(\int_0^t g(s)\mathrm{d}s\Big),$$

with $g$ the (unique) solution of the Volterra Ricatti equation

$$g(t) = \int_0^t f^{\alpha,\lambda}(t-s)\Big(\frac{\delta}{4}g(s)^2 + \frac{2}{\delta}f(s)\Big)\mathrm{d}s.$$

# The case $\alpha > 1/2$

## Rough Heston limit

When $\alpha > \frac{1}{2}$, the rescaled price process variance is almost surely differentiable. Furthermore

$$P_t = \int_0^t \sqrt{Y_s} \mathrm{d}B_s,$$

$$Y_t = \frac{1}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}\big(\frac{2}{\delta} - \lambda Y_s\big)\mathrm{d}s + \int_0^t (t-s)^{\alpha-1}\sqrt{Y_s}\mathrm{d}W_s\Big).$$

Therefore we have a rough Heston model with $H = \alpha - 1/2$.

# Summary

## From no-arbitrage to volatility

- We made two assumptions : Linear permanent impact and martingale price.
- Only modeling assumption : Hawkes dynamics for the order flow (reasonable...).
- This leads to rough volatility. In the square-root law case, $H \approx 0$.